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Schwarzschild Geometry Emerging from Matrix Models

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Abstract

We demonstrate how various geometries can emerge from Yang-Mills type matrix models with branes, and consider the examples of Schwarzschild and Reissner-Nordström geometry. We provide an explicit embedding of these branes in $\mathbb{R}^{2,5}$ and $\mathbb{R}^{4,6}$, as well as an appropriate Poisson resp. symplectic structure which determines the non-commutativity of space-time. The embedding is asymptotically flat with asymptotically constant $\theta^{\mu\nu}$ for large r, and therefore suitable for a generalization to many-body configurations. This is an illustration of our previous work [1], where we have shown how the Einstein-Hilbert action can be realized within such matrix models.

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1 Background and introduction

It has been argued in numerous publications that combining the basic aspects of Quantum Mechanics and General Relativity strongly suggests a quantum structure of space-time itself near the Planck scale — see for example Ref. [2]. One approach to realize this idea is to replace classical space-time by a quantized, or non-commutative (NC), space-time. Coordinate functions x^μ are promoted to Hermitian operators X^μ acting on a Hilbert space $\mathcal{H}$, which satisfy certain non-trivial commutation relations

$$[X^\mu, X^\nu] = i\theta^{\mu\nu}\,. \tag{1}$$

In the simplest case one may consider a Heisenberg algebra, corresponding to constant $\theta^{\mu\nu}$ which commutes with the X^μ. This has been studied extensively in the past (cf. [3–5] for a review of such "non-commutative" field theories). However, in the context of gravity it seems essential that this commutator $\theta^{\mu\nu}$ becomes dynamical. Indeed, semi-classically it determines a Poisson structure on space-time, as we will discuss below.

It has been shown previously [6–8] that matrix models of Yang-Mills type naturally realize this idea, and incorporate at least some version of (quantized) gravity; see Ref. [9] for a review. Hence we start our discussion with the matrix model action

$$S_{YM} = -\mathrm{Tr}[X^a, X^b][X^c, X^d]\eta_{ac}\eta_{bd}\,, \tag{2}$$

where η_{ab} denotes the (flat) metric of a D dimensional embedding space, with arbitrary signature. The "covariant coordinates" X^a are Hermitian matrices, resp. operators acting on a Hilbert space $\mathcal{H}$. It was shown in Ref. [10] that if one considers some of the coordinates to be functions of the remaining ones such that $X^a \sim x^a = (x^\mu, \phi^i(x^\mu))$ in the semi-classical limit, one can interpret the x^a as defining the embedding of a $2n$-dimensional submanifold $\mathcal{M}^{2n} \hookrightarrow \mathbb{R}^D$ equipped with a non-trivial induced metric

$$g_{\mu\nu}(x) = \partial_\mu x^a \partial_\nu x^b \eta_{ab}\,, \tag{3}$$

via pull-back of η_{ab}. In the present case we consider this submanifold to be a four dimensional space-time $\mathcal{M}^4$, and following [10] we can interpret

$$[X^\mu, X^\nu] \sim i\theta^{\mu\nu}(x) \tag{4}$$

as a Poisson structure on $\mathcal{M}^4$. Furthermore, we assume that $\theta^{\mu\nu}$ is non-degenerate, so that its inverse matrix $\theta^{-1}_{\mu\nu}$ defines a symplectic form $\Theta = \theta^{-1}_{\mu\nu} dx^\mu \wedge dx^\nu$ on $\mathcal{M}^4$.

The essential point is now that the Poisson structure $\theta^{\mu\nu}$ and the induced metric $g_{\mu\nu}$ combine to the effective metric

$$G^{\mu\nu} = e^{-\sigma}\theta^{\mu\rho}\theta^{\nu\sigma} g_{\rho\sigma}\,, \qquad\qquad e^{-\sigma} \equiv \frac{\sqrt{\det \theta^{-1}_{\mu\nu}}}{\sqrt{\det G_{\rho\sigma}}}\,. \tag{5}$$

It is, in fact, this effective metric $G_{\mu\nu}$ which is "seen" by matter [6] (i.e. scalar fields, gauge fields, and fermions possibly up to conformal factors), and which therefore must be interpreted in terms of gravity. In the present work, we restrict ourselves to the special case of $G_{\mu\nu} = g_{\mu\nu}$ in 4 dimensional space-time $\mathcal{M}^4$. It is easy to see that this is equivalent to $\theta^{-1}_{\mu\nu}$ being (anti)self-dual, by which in the case of Minkowski signature we mean $\star_g \Theta = \pm i\Theta$. This requires that $\theta^{\mu\nu}$ is complexified, as discussed in Section 2.2. The Yang-Mills action (2) then reduces in the semi-classical limit to

$$S_{YM} = -\mathrm{Tr}[X^a, X^b][X^c, X^d]\eta_{ac}\eta_{bd} \sim 4\int \sqrt{g}\,, \tag{6}$$

which in General Relativity (GR) is interpreted as cosmological constant. We also recall that (2) leads to the following equation of motion for $\theta^{\mu\nu}$

$$\nabla^\eta_G(e^\sigma \theta^{-1}_{\eta\nu}) = G_{\rho\nu}\,\theta^{\rho\mu} e^{-\sigma}\partial_\mu \eta\,, \qquad\qquad \eta \equiv \frac{e^\sigma}{4} G^{\mu\nu} g_{\mu\nu} = \Big|_{G=g} e^\sigma\,. \tag{7}$$

This equation holds identically for $G_{\mu\nu} = g_{\mu\nu}$ i.e. for self-dual $\theta^{\mu\nu}$, and is therefore not restricted to the model (2).

Einstein-Hilbert action. In a previous paper [1], we have shown that the following combination of higher order terms in the matrix model semi-classically lead to the Einstein-Hilbert type of action:

$$S_{\text{E-H}} = \mathrm{Tr}\left(2T^{ab}\Box X_a \Box X_b - T^{ab}\Box H_{ab}\right) \sim -2\int \sqrt{g}\, e^{2\sigma} R[g]\,, \tag{8}$$

where

$$\begin{aligned} T^{ab} &= \frac{1}{2}[[X^a, X^c], [X^b, X_c]]_+ - \frac{1}{4}\eta^{ab}[X^c, X^d][X_c, X_d]\,, \\ H^{ab} &= \frac{1}{2}[[X^a, X^c], [X^b, X_c]]_+\,, \\ \Box Y &\equiv [X^a, [X_a, Y]]\,. \end{aligned} \tag{9}$$

Latin indices are pulled down with the (flat) background metric η_{ab} (i.e. $X_a = \eta_{ab}X^b$), and $R[g]$ denotes the Ricci scalar with respect to the metric $G = g$ of the submanifold $\mathcal{M}^4$. Such actions can be added by hand, but they will also arise upon quantization of the Yang-Mills

matrix model (2). It was argued in [1] that the factor $e^{2\sigma}$ sets the scale and introduces the gravitational constant G.

Under reasonable conditions (such as global hyperbolicity), every 4-dimensional manifold can be equipped with a self-dual (complexified) symplectic form Θ. Then the classical embedding theorems [11, 12] imply that one can realize every 4-dimensional geometry as semi-classical configuration in the matrix model with $g_{\mu\nu} = G_{\mu\nu}$. In the present paper, we illustrate this general fact by providing an explicit construction of the most important solution: the Schwarzschild geometry. Subsequently, we also construct Reissner-Nordström (RN) geometry by following the same steps.

There are several possible actions which extend (8) beyond the case $g = G$ and which may imply different equations for $\theta^{\mu\nu}$ and for e^{σ}. We therefore restrict ourselves to the construction of geometries which are solutions to GR, equipped with self-dual $\theta^{\mu\nu}$. We do not check here in detail whether the above action (8) admits these spaces with self-dual $\theta^{\mu\nu}$ as solutions. Indeed additional terms in the action should be expected, leading e.g. to a potential for σ and possibly to deviations from $\theta^{\mu\nu}$ being self-dual. The point of this paper is not to present final answers but to illustrate how geometries such as Schwarzschild are expected to arise within this class of matrix models. In the same vein, we will also assume that the Yang-Mills resp. vacuum energy term (6) is negligible compared with the Einstein-Hilbert action (8), thus setting the cosmological constant to zero. There are several intriguing hints that the role of vacuum energy in this framework may be different than in GR [9].

Furthermore, we only consider the semi-classical limit of the matrix model in the present paper. Thus we will recover precisely the Schwarzschild geometry (resp. RN geometry), and the central singularity will be reflected by an embedding which escapes to infinity as one approaches the center. Of course, the main appeal for this framework compared with other descriptions of gravity is the fact that it goes beyond the classical concepts of geometry: Space-time is not put in by hand but *emerges*, realized as non-commutative space with an effective geometry, along with gauge fields and matter. Hence one should expect that non-commutative modifications become important as one approaches the singularity. However, this requires to go beyond the semi-classical approximations of this paper, which we will indicate by briefly discussing higher-order terms in the star product in Appendix B.

Finally, we want to emphasize that the actions under consideration are expected to arise upon quantization of Yang-Mills matrix models, such as the IKKT model [13]. In particular the latter model is a promising candidate for a *quantum* theory of fundamental interactions including gravity. Of course, much more work remains to be done in order to fully understand this class of models.

2 The Schwarzschild geometry

We now show how the most important solution of General Relativity can emerge from the class of extended matrix model action presented in the previous section: the Schwarzschild geometry. We will restrict ourselves to the semi-classical limit here, however a possible way to obtain higher-order corrections in $\theta^{\mu\nu}$ is discussed in Appendix B.

2.1 Embedding of Schwarzschild geometry

Our construction involves two steps:

1) the choice of a suitable embedding $\mathcal{M}^4 \subset \mathbb{R}^D$ such that the induced geometry on $\mathcal{M}^4$ given by $g_{\mu\nu}$ is the Schwarzschild metric, and
2) a suitable non-degenerate Poisson structure on $\mathcal{M}^4$ which solves the e.o.m. $\nabla^\mu \theta^{-1}_{\mu\nu} = 0$ for self-dual symplectic form Θ.

Both steps are far from unique a priori. However, the freedom is considerably reduced by requiring that the solution should be a "local perturbation" of an asymptotically flat (or nearly flat) "cosmological" background. This is clear on physical grounds, having in mind the geometry near a star in some larger cosmological context: it must be possible to approximately "superimpose" our solution, allowing e.g. for systems of stars and galaxies in a natural way. This eliminates the well-known embeddings of the Schwarzschild geometry in the literature [14–16], which are highly non-trivial for large r and cannot be superimposed in any obvious way. In fact we require that the embedding is asymptotically harmonic $\Box x^a \to 0$ for $r \to \infty$, in view of the fact that there may be terms in the matrix model which depend on the extrinsic geometry, and which typically single out such harmonic embeddings[1].

Furthermore, we insist that $\theta^{\mu\nu}$ is non-degenerate, and $\theta^{\mu\nu} \to \text{const.} \neq 0$ as $r \to \infty$. This is again motivated by the requirement that physics at large distances should not be affected by a localized mass. In particular, e^σ defines essentially the scale of non-commutativity, and certainly enters in some way e.g. the physics of elementary particles (In fact, e^σ determines the strength of the gauge coupling in the matrix model [6, 10]). Therefore, e^σ should be asymptotically constant and non-vanishing. This is an important difference to previous proposals for a non-commutative Schwarzschild geometry (see in particular [18, 19] and references therein), where the Poisson structure is degenerate and/or not asymptotically constant. Hence $\theta^{\mu\nu}$ will be viewed as some cosmological background field which is locally perturbed by a mass. Recall that such a background is essentially invisible, since there are no fields in the matrix model which are charged under the corresponding $U(1)$. It enters the effective actions only through the gravitational metric $G_{\mu\nu}$.

Note that these boundary conditions for $\theta^{\mu\nu}$ are not in conflict with the idea that $\theta^{\mu\nu}$ may be locally fluctuating and should perhaps be averaged over or integrated out. We will discuss this possibility further below.

In order to obtain an appropriate embedding, keeping in mind the conditions we have just discussed, we consider Eddington-Finkelstein coordinates and define:

$$t = t_S + (r^* - r)\,, \qquad\qquad r^* = r + r_c \ln\left|\frac{r}{r_c} - 1\right|\,, \tag{10}$$

where t_S denotes the usual Schwarzschild time, r_c is the horizon of the Schwarzschild black hole and r^* is the well-known tortoise coordinate. The metric in Eddington-Finkelstein coordinates $\{t, r, \vartheta, \varphi\}$ is

$$ds^2 = -\left(1 - \frac{r_c}{r}\right)dt^2 + \frac{2r_c}{r}dtdr + \left(1 + \frac{r_c}{r}\right)dr^2 + r^2 d\Omega^2 \tag{11}$$

which is asymptotically flat for large r, and manifestly regular at the horizon r_c. Thus, we only need to find a good embedding of this metric. To reproduce the mixed term, consider first

$$\phi_1 + i\phi_2 = h(r)e^{i(\omega t + g(r))} \tag{12}$$

[1]This can hold only asymptotically, since Ricci-flat geometries can in general *not* be embedded harmonically [17].

which satisfies

$$\begin{aligned}
\partial_t\phi^i\partial_t\phi^i &= \omega^2 h^2\,,\\
\partial_t\phi^i\partial_r\phi^i &= \omega g' h^2\,,\\
\partial_r\phi^i\partial_r\phi^i &= g'^2 h^2 + h'^2\,.
\end{aligned} \tag{13}$$

So we demand

$$\omega g' h^2 = \frac{r_c}{r} = \omega^2 h^2 \tag{14}$$

which is satisfied for

$$h(r) = \frac{1}{\omega}\sqrt{\frac{r_c}{r}}\,, \qquad\qquad g(r) = \omega r\,. \tag{15}$$

Furthermore, since $g'^2h^2 = \omega g' h^2 = \frac{r_c}{r}$, we need to cancel the h'^2 term in Eqn. (13) above. Hence, we need another coordinate

$$\phi_3 = h(r), \tag{16}$$

with time-like embedding. So we have

$$\begin{aligned}
\phi_1 + i\phi_2 &= \phi_3 e^{i\omega(t+r)}\,,\\
\phi_3 &= \frac{1}{\omega}\sqrt{\frac{r_c}{r}}\,,
\end{aligned} \tag{17}$$

and the embedding of $\mathcal{M}^4 \subset \mathbb{R}^7$ is given by

$$x^a = \begin{pmatrix} t \\ r\cos\varphi\sin\vartheta \\ r\sin\varphi\sin\vartheta \\ r\cos\vartheta \\ \frac{1}{\omega}\sqrt{\frac{r_c}{r}}\cos\left(\omega(t+r)\right) \\ \frac{1}{\omega}\sqrt{\frac{r_c}{r}}\sin\left(\omega(t+r)\right) \\ \frac{1}{\omega}\sqrt{\frac{r_c}{r}} \end{pmatrix}\,, \tag{18}$$

(i.e. we consider $D = 7$ in this example). Together with the background metric

$$\eta_{ab} = \mathrm{diag}(-,+,+,+,+,+,-)\,, \tag{19}$$

this induces precisely the Eddington-Finkelstein metric (11) above.

Before we proceed to determine the symplectic form, let us take a closer look at the properties of this embedding: First, notice that the ω (appearing in the ϕ_i) does not enter the effective four dimensional metric, i.e. it is "hidden" in the three extra dimensions. Furthermore, we must emphazise that ϕ_3 is an additional *time-like* direction. Asymptotically, i.e. for $r \to \infty$, Eqn. (18) describes flat four dimensional Minkowski space where the extra dimensions $\phi_i \sim \frac{1}{r}$ become infinitesimally small. On the other hand, when one approaches the singularity of the Schwarzschild black hole at $r = 0$, these extra dimensions blow up and become arbitrarily large. In particular, note that then ϕ_3 should be interpreted as asymptotic time (which is unbounded), i.e.

$$T := \phi_3\,, \qquad r = r_c\,\frac{1}{\omega^2 T^2}\,, \tag{20}$$

so that

$$x^a = \begin{pmatrix} x^0 \\ x^i \\ \phi_1 + i\phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} t, \\ r_c \, \frac{1}{\omega^2 T^2} \left(\sin\vartheta \sin\varphi, \sin\vartheta\cos\varphi, \cos\vartheta\right) \\ T\, e^{i\omega(t + r_c\, \frac{1}{\omega^2 T^2})} \\ T \end{pmatrix} \tag{21}$$

for large T. This is a helicoid-like (cone-like) geometry in $\{T, t\}$ with (increasing) radius T and t playing the role of the angle variable, times a contracting sphere of radius $\frac{1}{T^2}$. The geometry of the submanifold $\mathcal{M}^{1,3} \subset \mathbb{R}^{2,5}$ is completely regular, and the central singularity is reflected by an embedding which escapes to infinity. Near this singularity, the geometry is effectively 2-dimensional. An illustration is given by Figure 1.

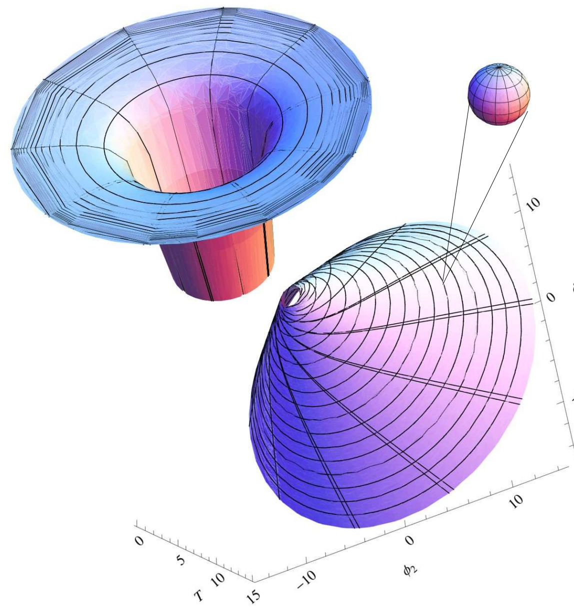

Figure 1: Embedded Schwarzschild black hole. *On the top, a schematic view of the outer region of the Schwarzschild black hole is shown. After passing through the horizon $r = r_c$, the extra dimensions ϕ_i "blow up" in a cone-like manner. As indicated in the lower half of this figure, every point of the cone is in fact a sphere whose radius r becomes smaller towards the bottom of the cone (i.e. $T \propto 1/\sqrt{r}$). The twisted vertical lines drawn in the cone are lines of equal time t.*

Of course, quantum effects will play a major role near $r = 0$. This implies that the semi-classical approximation we are currently considering will break down in the vicinity of that region. We expect that these non-commutative effects will regularize the would-be singularity. For example, the contracting sphere of radius $\frac{1}{T^2}$ may become fuzzy [20], so that for large T the present geometry could become effectively 2-dimensional with an extra-dimensional fuzzy sphere.

In order to understand the meaning of the extra dimensions ϕ_i at large distances $r \to \infty$ from the Schwarzschild black hole, it is instructive to consider the following modification resp. higher-dimensional extension of the Schwarzschild geometry. Consider the 6-dimensional space

$\mathbb{R}^4 \times AdS^2 \subset \mathbb{R}^7$ defined by

$$\phi_1^2 + \phi_2^2 - \phi_3^2 = R^2 \,. \tag{22}$$

Here the ϕ_i describe an AdS^2 space embedded in $\mathbb{R}^3$, which can be parametrized as

$$\begin{aligned} \phi_1 + i\phi_2 &= \sqrt{\phi_3^2 + R^2}\, e^{i\omega u} \,, \\ \phi_3 &= \phi_3 \,. \end{aligned} \tag{23}$$

The Schwarzschild manifold described above is then recovered by setting

$$u = t + r \,, \qquad \phi_3 = \frac{1}{\omega}\sqrt{\frac{r_c}{r}} \,, \quad \text{and } R = 0 \,, \tag{24}$$

while $R \neq 0$ corresponds to a modification of the Schwarzschild geometry. The length element of $AdS^2 \subset \mathbb{R}^3$ is given by

$$ds^2 = d\phi_1^2 + d\phi_2^2 - d\phi_3^2 = \omega^2 \left(\phi_3^2 + R^2\right) du^2 - \frac{R^2}{\phi_3^2 + R^2} d\phi_3^2 \,. \tag{25}$$

Note that there is no contribution from the time-like coordinate $d\phi_3^2$ for $R = 0$ since it is embedded in a null direction. The metric on AdS^2 then becomes degenerate and space-like, with very small radius as $r \to \infty$. Hence, this extra AdS^2 can be interpreted as physical extra dimension which naturally becomes "invisible" for large r, i.e. far away from the Schwarzschild black hole. The point is that such an AdS^2 could arise naturally in matrix models similar to fuzzy spheres, and may play an interesting physical role, cf. [21–24].

2.2 Symplectic form

As mentioned in the introduction we consider the simple class of self-dual geometries where the effective metric $G_{\mu\nu}$ equals the induced metric $g_{\mu\nu}$. Hence we need to find an (anti)self-dual symplectic form Θ so that

$$\begin{aligned} G^{\mu\nu} &= e^{\sigma} \theta^{\mu\rho} \theta^{\nu\sigma} g_{\rho\sigma} \\ &= g^{\mu\nu} \,. \end{aligned} \tag{26}$$

At this point, we recall that

$$\mathcal{J}_\gamma^\eta = e^{-\sigma/2}\, \theta^{\eta\gamma'} g_{\gamma'\gamma} \tag{27}$$

satisfies

$$\mathcal{J}^2 = -1 \quad \Leftrightarrow \quad \star\Theta = \pm i\Theta \quad \Leftrightarrow \quad g_{\mu\nu} = G_{\mu\nu} \,, \tag{28}$$

so that we are dealing with an almost complex manifold. Moreover, the symplectic structure is necessarily complexified in a way which is determined by $\mathcal{J}^2 = -1$. Thus the last relation specifies[2] the "real form" of $\theta^{\mu\nu}$.

Furthermore, we require $\Theta = \theta^{-1}_{\mu\nu} dx^\mu \wedge dx^\nu$ to lead to an asymptotically constant $e^{-\sigma}$ since, as mentioned previously, we would like to describe everything as a local perturbation of flat Moyal space. To be more specific, we demand

$$\lim_{r\to\infty} e^{-\sigma} = \text{const.} \neq 0 \,. \tag{29}$$

[2]In the case of general geometries $g_{\mu\nu} \neq G_{\mu\nu}$ this is replaced by a quartic relation for $\mathcal{J}$ [25].

In order to find such a symplectic form Θ, we consider the following: The Schwarzschild metric has two Killing vector fields $V_{ts} = \partial_{t_S}$ and $V_\varphi = \partial_\varphi$. Hence, in Schwarzschild coordinates this leads to the ansatz

$$\begin{aligned}
\Theta &= i\Theta_E + \Theta_B = iE \wedge dt_S + \Theta_B, \\
E &= i_{V_{ts}}\Theta_E = E_r dr + E_\vartheta d\vartheta + E_\varphi d\varphi, \\
\Theta_B &= B_r d\vartheta \wedge d\varphi + B_\vartheta dr \wedge d\varphi + B_\varphi d\vartheta \wedge dr \\
&= \star\Theta_E \,,
\end{aligned} \tag{30}$$

which implements self-duality, i.e.

$$\star\Theta = i\Theta \,, \tag{31}$$

supplemented by the conditions

$$\begin{aligned}
\mathcal{L}_{V_{ts}}\Theta &= 0\,, \\
\mathcal{L}_{V_\varphi}\Theta &= 0\,.
\end{aligned} \tag{32}$$

A solution which satisfies the required asymptotics (29) is then given by

$$\begin{aligned}
E &= c_1 \left(\cos\vartheta dr - r(1-\frac{r_c}{r})\sin\vartheta d\vartheta\right) = d(f(r)\cos\vartheta)\,, \\
B &= c_1 \left(r^2 \sin\vartheta\cos\vartheta d\vartheta + r\sin^2\vartheta dr\right) = \frac{c_1}{2} d(r^2\sin^2\vartheta)\,, \\
\Theta &= iE\wedge dt_S + B\wedge d\varphi\,, \\
&\text{with} \quad f(r) = c_1 r(1-\frac{r_c}{r}), \qquad f' = c_1 = \text{const.}\,,
\end{aligned} \tag{33}$$

from which one finds

$$e^{-\sigma} = c_1^2\left(1-\frac{r_c}{r}\sin^2\vartheta\right)\,. \tag{34}$$

Details of the computation are given in Appendix A. This can be interpreted as a (complexified) electromagnetic field with asymptotically constant fields E, B pointing in the z direction, and $e^{-\sigma}$ is indeed asymptotically constant. Other solutions are of course obtained by acting with the rotation group on the asymptotic E resp. B field.

Note in particular that we have obtained metric-compatible Darboux coordinates (resp. Hamiltonian reduction) $x_D^\mu = \{H_{ts}, t_S, H_\varphi, \varphi\}$ corresponding to V_{ts}, V_φ where the symplectic form is constant:

$$\begin{aligned}
\Theta &= ic_1 dH_{ts}\wedge dt_S + c_1 dH_\varphi \wedge d\varphi\,, \\
&= c_1 d\left(iH_{ts}dt_S + H_\varphi d\varphi\right)\,, \\
H_{ts} &= r\cos\vartheta(1-\frac{r_c}{r})\,, \qquad H_\varphi = \frac{1}{2}r^2\sin^2\vartheta\,.
\end{aligned} \tag{35}$$

The Schwarzschild metric in Darboux coordinates reads

$$ds^2 = -\left(1-\frac{r_c}{r}\right)dt_S^2 + \frac{e^{\bar\sigma}}{(1-\frac{r_c}{r})}dH_{ts}^2 + r^2\sin^2\vartheta d\varphi^2 + \frac{e^{\bar\sigma}}{r^2\sin^2\vartheta}dH_\varphi^2\,, \tag{36}$$

with $e^{\bar{\sigma}} = c_1^2 e^{\sigma} = \left(1 - \frac{r_c}{r}\sin^2\vartheta\right)^{-1}$. Notice that no $dH_{ts}dH_{\varphi}$-term appears, i.e. the two Darboux blocks do not mix. The relations to the Killing vector fields are:

$$
\begin{aligned}
E &= c_1 dH_{ts} = c_1 E_\mu dx^\mu = i_{V_{ts}}\Theta\,, & E_\mu &= V_{ts}^\nu \theta_{\nu\mu}^{-1}\,, \\
B &= c_1 dH_{\varphi} = c_1 B_\mu dx^\mu = i_{V_{\varphi}}\Theta\,, & B_\mu &= V_{\varphi}^\nu \theta_{\nu\mu}^{-1}\,,
\end{aligned}
\tag{37}
$$

(cf. Eqn. (33) above). In order to obtain the Poisson brackets between the Cartesian matrix coordinates, we will transform t_S to Eddington-Finkelstein time t and invert it so as to derive $\theta^{\mu\nu}$. Subsequently, we will extend the θ-matrix to the seven dimensional embedding space of Eqn. (18) as that will provide us with the leading order commutator relations between the coordinates, i.e. $[X^a, X^b] \sim i\theta^{ab}$. As shown in Appendix B, this leads to the following semi-classical commutation relations for the 7-dimensional coordinates $X^a \sim x^a = \{t, x, y, z, \phi_1, \phi_2, \phi_3\}$:

$$
\theta^{ab} = \epsilon e^{\bar{\sigma}} \begin{pmatrix}
0 & -\frac{r_c y}{r^2} & \frac{r_c x}{r^2} & -i & \frac{iz}{r} f_{12}^+(0) & \frac{iz}{r} f_{21}^-(0) & \frac{iz\phi_3}{2r^2} \\
\frac{r_c y}{r^2} & 0 & e^{-\bar{\sigma}} & -\frac{r_c yz}{r^3} & -\frac{y}{r} f_{12}^+(r_c) & -\frac{y}{r} f_{21}^-(r_c) & -\frac{y\gamma\phi_3}{2r^2} \\
-\frac{r_c x}{r^2} & -e^{-\bar{\sigma}} & 0 & \frac{r_c xz}{r^3} & \frac{x}{r} f_{12}^+(r_c) & \frac{x}{r} f_{21}^-(r_c) & \frac{x\gamma\phi_3}{2r^2} \\
i & \frac{r_c yz}{r^3} & -\frac{r_c xz}{r^3} & 0 & -i\omega\phi_2 & i\omega\phi_1 & 0 \\
-\frac{iz}{r} f_{12}^+(0) & \frac{y}{r} f_{12}^+(r_c) & -\frac{x}{r} f_{12}^+(r_c) & i\omega\phi_2 & 0 & -\frac{i\omega z\phi_3^2}{2r^2} & -\frac{i\omega z\phi_3\phi_2}{2r^2} \\
-\frac{iz}{r} f_{21}^-(0) & \frac{y}{r} f_{21}^-(r_c) & -\frac{x}{r} f_{21}^-(r_c) & -i\omega\phi_1 & \frac{i\omega z\phi_3^2}{2r^2} & 0 & \frac{i\omega z\phi_3\phi_1}{2r^2} \\
-\frac{iz\phi_3}{2r^2} & \frac{y\gamma\phi_3}{2r^2} & -\frac{x\gamma\phi_3}{2r^2} & 0 & \frac{i\omega z\phi_3\phi_2}{2r^2} & -\frac{i\omega z\phi_3\phi_1}{2r^2} & 0
\end{pmatrix},
\tag{38}
$$

with

$$
\begin{aligned}
f_{ij}^{\pm}(r_c) &= \left(\frac{\gamma}{2r}\phi_i \pm \omega\phi_j\right)\,, \\
\gamma &= \left(1 - \frac{r_c}{r}\right)\,, \\
e^{-\bar{\sigma}} &= \frac{e^{-\sigma}}{c_1^2} = \epsilon^2 e^{-\sigma}\,.
\end{aligned}
\tag{39}
$$

This defines a Poisson structure on $\mathcal{M}^4$, but it could also be viewed as a Poisson structure on the 6-dimensional space defined by $\phi_1^2 + \phi_2^2 = \phi_3^2$ which admits $\mathcal{M}^4$ as symplectic leaf. As a consistency check, the interested reader may verify that relation (26) is indeed fulfilled (on the 4-dimensional submanifold $\mathcal{M}^4$), and that the Jacobi identity holds as well.

2.3 Star product

So far, we have worked only in the semi-classical limit. In order to see some effects of the space-time quantization, we may for instance compute the next-to-leading order commutation relations. For this purpose, recall the Darboux coordinates $x_D^\mu = \{t_S, H_{ts}, \varphi, H_\varphi\}$ we derived in (35). Since in these coordinates the Poisson structure $\theta^{\mu\nu}$ (of the 4 dim. submanifold $\mathcal{M}^4$) is constant, we can easily define a Moyal-type star product [3, 4] as

$$
(g \star h)(x_D) = g(x_D) e^{-\frac{i}{2}\left(\overleftarrow{\partial}_\mu \theta_D^{\mu\nu} \overrightarrow{\partial}_\nu\right)} h(x_D)\,,
\tag{40}
$$

with

$$
\theta_D^{\mu\nu} = \epsilon \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}\,,
\tag{41}
$$

where $\epsilon = 1/c_1 \ll 1$ denotes the expansion parameter. In order to derive a star product in terms of the Cartesian coordinates $x^\mu = \{t, x, y, z\}$, all we need is the coordinate transformation (76) of Appendix B leading to

$$\begin{aligned}(g \star h)(x) = g(x) \exp\Bigg[\frac{i\epsilon}{2}\Bigg(&\left(\overleftarrow{\partial}_t \frac{i r_c z e^{\bar\sigma}}{r(r-r_c)} + \overleftarrow{\partial}_z i e^{\bar\sigma}\right) \wedge \overrightarrow{\partial}_t \\ &+ \left(\left(\overleftarrow{\partial}_t - \overleftarrow{\partial}_z \frac{z}{r}\right)\frac{r_c e^{\bar\sigma}}{r^2} + \left(\overleftarrow{\partial}_x x + \overleftarrow{\partial}_y y\right)\frac{1}{x^2+y^2}\right) \wedge \left(x \overrightarrow{\partial}_y - y \overrightarrow{\partial}_x\right)\Bigg)\Bigg] h(x)\,,\end{aligned} \tag{42}$$

where the wedge stands for "antisymmetrized", and when considering the expansion one must take care with the sequence of operators and the side they act on (left or right). One can then compute next-to-leading order contributions to the commutation relations (38). Some of the relations can be computed to all orders[3], i.e.

$$[t \stackrel{\star}{,} z] = \epsilon e^{\bar\sigma}\,, \qquad [x \stackrel{\star}{,} y] = i\epsilon\,,$$
$$[t \stackrel{\star}{,} \phi_3] = -\epsilon e^{\bar\sigma}\frac{z\phi_3}{2r^2}\,, \qquad [z \stackrel{\star}{,} \phi_3] = 0\,, \tag{43}$$

while the others receive corrections — see Eqns. (78)-(83) in Appendix B for the full expressions. Hence, also the embedding constraint $\phi_1^2 + \phi_2^2 = \phi_3^2$ is modified under the star product, i.e. we have

$$\begin{aligned}\frac{1}{2}\left[\phi_1 + i\phi_2 \stackrel{\star}{,} \phi_1 - i\phi_2\right]_+ &= \phi_1 \star \phi_1 + \phi_2 \star \phi_2 \\ &= \phi_3^2 + \epsilon^2 \phi_3^2 \frac{\omega^2 e^{2\bar\sigma}}{8r^2}\left(1 - 3\frac{e^{\bar\sigma} z^2}{r^2}\right) + \mathcal{O}\left(\epsilon^4\right)\,,\end{aligned} \tag{44}$$

while $\phi_3 \star \phi_3 = \phi_3^2$ to all orders. This could be interpreted as non-commutative correction to the embedding geometry.

3 The Reissner-Nordström geometry

In this section, we continue by presenting the semi-classical quantization of another geometry: the Reissner-Nordström (RN) geometry.

3.1 Embedding of the geometry

We start by considering the usual RN metric in spherical coordinates $x^\mu = \{t, r, \vartheta, \varphi\}$:

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) d\tilde t^2 + \frac{1}{\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)} dr^2 + r^2 d\Omega\,, \tag{45}$$

[3] It is also interesting to note, that the quantities $\{z,\ \phi_3,\ H_{\vartheta_2}\}$, where

$$H_\varphi = \tfrac{1}{2}x_+ x_- = \tfrac{1}{2}(x^2+y^2) = \tfrac{1}{4}\left[x_+ \stackrel{\star}{,} x_-\right]_+\,, \qquad x_\pm = x \pm iy\,,$$

commute with each other to all orders in ϵ.

where m denotes the mass and q is the charge of the black hole. This geometry has two concentric horizons at

$$r_h = \left(m \pm \sqrt{m^2 - q^2}\right) \,, \tag{46}$$

and in the following, we assume that $q^2 < m^2$. In order to transform this metric into coordinates which are similar to Eddington-Finkelstein, we consider radial null geodesics. These are given by

$$\begin{aligned} 0 &= -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)\left(d\tilde{t}^2 - \frac{1}{\left(1 - \frac{2m}{r} - \frac{q^2}{r^2}\right)^2} dr^2\right) \\ &\equiv -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)\left(d\tilde{t}^2 - (dr^*)^2\right) \,, \end{aligned} \tag{47}$$

defining the tortoise-like coordinate r^*. The in and outgoing geodesics are $V = \tilde{t} + r^*$ and $U = \tilde{t} - r^*$. Explicitly, we have

$$r^* = r + m \ln \left|r^2 - 2mr + q^2\right| + \frac{2m^2 - q^2}{2\sqrt{m^2 - q^2}} \ln \left| \frac{\sqrt{m^2 - q^2} - (r - m)}{\sqrt{m^2 - q^2} + (r - m)} \right| \,. \tag{48}$$

As in the Schwarzschild case, we use this coordinate to shift the time-coordinate according to

$$t = \tilde{t} + (r^* - r) \,, \tag{49}$$

and arrive at the transformed RN metric

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) dt^2 + 2\left(\frac{2m}{r} - \frac{q^2}{r^2}\right) dt dr + \left(1 + \frac{2m}{r} - \frac{q^2}{r^2}\right) dr^2 + r^2 d\Omega \,. \tag{50}$$

Observe, that the metric (50) has exactly the same form as the Eddington-Finkelstein metric (11) of Schwarzschild geometry, but with the replacement

$$\frac{r_c}{r} \to \frac{2m}{r} - \frac{q^2}{r^2} \,. \tag{51}$$

Hence, motivated by the Schwarzschild geometry case, we can use the 10-dimensional embedding $\mathcal{M}^{1,3} \hookrightarrow \mathbb{R}^{4,6}$ with the additional coordinates ϕ_i given by

$$\begin{aligned} \phi_1 + i\phi_2 &= \phi_3 e^{i\omega(t+r)} \,, & \phi_3 &= \frac{1}{\omega}\sqrt{\frac{2m}{r}} \,, \\ \phi_4 + i\phi_5 &= \phi_6 e^{i\omega(t+r)} \,, & \phi_6 &= \frac{q}{\omega r} \,. \end{aligned} \tag{52}$$

Note that ϕ_3, ϕ_4 and ϕ_5 are *time-like* coordinates, i.e. we consider the background metric

$$\eta_{ab} = \mathrm{diag}(-, +, +, +, +, +, -, -, -, +) \,. \tag{53}$$

Like in the previous case, ω does not enter the induced metric (50), but is hidden in the extra dimensions ϕ_i. For $r \to \infty$, the ϕ_i become infinitesimally small and hence asymptotically, the four dimensional subspace becomes flat Minkowski space-time.

3.2 Symplectic form

A self-dual symplectic form Θ can be computed in the same way as in the Schwarzschild case leading to metric compatible Darboux coordinates $x_D^\mu = \{H_{\tilde{t}}, \tilde{t}, H_\varphi, \varphi\}$ with

$$\begin{aligned}
H_{\tilde{t}} &= r\gamma\cos\vartheta = z\gamma\,, \qquad\qquad H_\varphi = \alpha\frac{r^2}{2}\sin^2\vartheta = \alpha\frac{x^2+y^2}{2}\,,\\
\Theta &= idH_{\tilde{t}}\wedge d\tilde{t} + dH_\varphi\wedge\varphi\,,\\
e^{-\bar{\sigma}} &= \gamma\sin^2\vartheta + \alpha^2\cos^2\vartheta = \gamma\left(1-\frac{q^2z^2}{r^4}\right) + \alpha\eta\frac{z^2}{r^2}\,,\\
\gamma &= \left(1-\frac{2m}{r}+\frac{q^2}{r^2}\right)\,, \qquad \alpha = \left(1-\frac{q^2}{r^2}\right)\,, \qquad \eta = 2\left(\frac{m}{r}-\frac{q^2}{r^2}\right)\,,
\end{aligned} \tag{54}$$

and the RN metric in Darboux coordinates reads

$$ds^2 = -\gamma d\tilde{t}^2 + \frac{e^{\bar{\sigma}}}{\gamma}dH_{\tilde{t}}^2 + r^2\sin^2\vartheta d\varphi^2 + \frac{e^{\bar{\sigma}}}{r^2\sin^2\vartheta}dH_\varphi^2\,, \tag{55}$$

a form similar to the according Schwarzschild metric (36). In the limit $q\to 0$ these expressions reduce to those in the Schwarzschild case. Furthermore, one can easily check that $\star\Theta = i\Theta$ and $G_{\mu\nu} = g_{\mu\nu}$.

3.3 Star product

A Moyal type star product can easily be defined in Darboux coordinates as

$$(g\star h)(x_D) = g(x_D)e^{-\frac{i}{2}\left(\overleftarrow{\partial}_\mu\theta_D^{\mu\nu}\overrightarrow{\partial}_\nu\right)}h(x_D)\,, \tag{56}$$

with

$$\theta_D^{\mu\nu} = \epsilon\begin{pmatrix} 0 & i & 0 & 0\\ -i & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{pmatrix}\,, \qquad \epsilon\in\mathbb{R}\,. \tag{57}$$

Transforming these Darboux coordinates back to the Cartesian ones, where

$$t = \tilde{t} + (r^* - r)\,, \qquad r^2 = x^2+y^2+z^2\,, \tag{58}$$

we eventually find

$$\partial_t = \partial_{\tilde{t}}\,, \qquad \partial_\varphi = -y\partial_x + x\partial_y\,, \tag{59a}$$

and

$$\begin{aligned}
\partial_{H_{\tilde{t}}} &= e^{\bar{\sigma}}\left[\alpha\frac{z}{r}\left(\frac{1}{\gamma}-1\right)\partial_t + \left(1-\frac{q^2z^2}{r^4}\right)\partial_z - \frac{q^2z}{r^4}\left(x\partial_x+y\partial_y\right)\right]\,,\\
\partial_{H_\varphi} &= \frac{e^{\bar{\sigma}}}{r}\left[(\gamma-1)\,\partial_t - \eta\frac{z}{r}\partial_z\right] + \frac{e^{\bar{\sigma}-\varsigma}}{x^2+y^2}\left(x\partial_x+y\partial_y\right)\,,\\
e^{-\varsigma} &= \left(\gamma+\eta\frac{z^2}{r^2}\right)\,,
\end{aligned} \tag{59b}$$

(cf. the abbreviations defined in Eqn. (54)). The star product in Cartesian coordinates hence reads

$$(g\star h)(x) = g(x)\exp\left[\frac{i\epsilon}{2}\left(i\overleftarrow{\partial}_{H_{\tilde{t}}}\wedge\overrightarrow{\partial}_{t}+\overleftarrow{\partial}_{H_{\varphi}}\wedge\overrightarrow{\partial}_{\varphi}\right)\right]h(x)\,, \tag{60}$$

with Eqn. (59a) and Eqn. (59b), where once more the wedge stands for "antisymmetrized", and when considering the expansion one must take care with the sequence of operators and the side they act on (left or right). The first order results for the star commutators between the 10-dimensional embedding coordinates are given by Eqns. (61):

$$-i\left[x^{\mu}\overset{\star}{,}x^{\nu}\right]\approx\theta^{\mu\nu}=\epsilon e^{\bar{\sigma}}\begin{pmatrix} 0 & \frac{-(1-\gamma)y}{r}+\frac{iq^2xz}{r^4} & \frac{(1-\gamma)x}{r}+\frac{iq^2yz}{r^4} & -i\beta \\ \frac{(1-\gamma)y}{r} & 0 & e^{-\varsigma} & \frac{-yz\eta}{r^2} \\ \frac{-(1-\gamma)x}{r} & -e^{-\varsigma} & 0 & \frac{xz\eta}{r^2} \\ i\beta & \frac{yz\eta}{r^2} & \frac{-xz\eta}{r^2} & 0 \end{pmatrix}, \tag{61a}$$

$$-i\left[\phi_i\overset{\star}{,}x^{\mu}\right]\approx\epsilon e^{\bar{\sigma}}\begin{pmatrix} \frac{-iz\alpha f^{+}_{12}\left(\frac{1}{2}\right)}{r} & \frac{yf^{+}_{12}\left(\frac{\gamma}{2}\right)}{r}-\frac{iq^2xz\omega\phi_2}{r^4} & \frac{-xf^{+}_{12}\left(\frac{\gamma}{2}\right)}{r}-\frac{iq^2yz\omega\phi_2}{r^4} & i\omega\phi_2\beta \\ \frac{-iz\alpha f^{-}_{21}\left(\frac{1}{2}\right)}{r} & \frac{yf^{-}_{21}\left(\frac{\gamma}{2}\right)}{r}+\frac{iq^2xz\omega\phi_1}{r^4} & \frac{-xf^{-}_{21}\left(\frac{\gamma}{2}\right)}{r}+\frac{iq^2yz\omega\phi_1}{r^4} & -i\omega\phi_1\beta \\ \frac{-iz\phi_3\alpha}{2r^2} & \frac{y\gamma\phi_3}{2r^2} & \frac{-x\gamma\phi_3}{2r^2} & 0 \\ \frac{-iz\alpha f^{+}_{45}(1)}{r} & \frac{yf^{+}_{45}(\gamma)}{r}-\frac{iq^2xz\omega\phi_5}{r^4} & \frac{-xf^{+}_{45}(\gamma)}{r}-\frac{iq^2yz\omega\phi_5}{r^4} & i\omega\phi_5\beta \\ \frac{-iz\alpha f^{-}_{54}(1)}{r} & \frac{yf^{-}_{54}(\gamma)}{r}+\frac{iq^2xz\omega\phi_4}{r^4} & \frac{-xf^{-}_{54}(\gamma)}{r}+\frac{iq^2yz\omega\phi_4}{r^4} & -i\omega\phi_4\beta \\ \frac{-iz\phi_6\alpha}{r^2} & \frac{y\gamma\phi_6}{r^2} & \frac{-x\gamma\phi_6}{r^2} & 0 \end{pmatrix}, \tag{61b}$$

$$-i\left[\phi_i\overset{\star}{,}\phi_j\right]\approx\epsilon e^{\bar{\sigma}}\begin{pmatrix} 0 & \frac{-i\omega z\phi_3^2\alpha}{2r^2} & \frac{-i\omega z\phi_3\phi_2\alpha}{2r^2} & \frac{-i\omega z\phi_1\phi_5\alpha}{2r^2} & \frac{-i\omega z\alpha g_\phi}{2r^2} & \frac{-i\omega z\phi_3\phi_5\alpha}{r^2} \\ \frac{i\omega z\phi_3^2\alpha}{2r^2} & 0 & \frac{i\omega z\phi_3\phi_1\alpha}{2r^2} & \frac{-i\omega z\alpha g_\phi}{2r^2} & \frac{i\omega z\phi_2\phi_4\alpha}{2r^2} & \frac{i\omega z\phi_3\phi_4\alpha}{r^2} \\ \frac{i\omega z\phi_3\phi_2\alpha}{2r^2} & \frac{-i\omega z\phi_3\phi_1\alpha}{2r^2} & 0 & \frac{i\omega z\phi_3\phi_5\alpha}{2r^2} & -\frac{i\omega z\phi_3\phi_4\alpha}{2r^2} & 0 \\ \frac{i\omega z\phi_1\phi_5\alpha}{2r^2} & \frac{i\omega z\alpha g_\phi}{2r^2} & \frac{-i\omega z\phi_3\phi_5\alpha}{2r^2} & 0 & \frac{-i\omega z\phi_6^2\alpha}{r^2} & \frac{-i\omega z\phi_5\phi_6\alpha}{r^2} \\ \frac{i\omega z\alpha g_\phi}{2r^2} & \frac{-i\omega z\phi_2\phi_4\alpha}{2r^2} & \frac{i\omega z\phi_3\phi_4\alpha}{2r^2} & \frac{i\omega z\phi_6^2\alpha}{r^2} & 0 & \frac{i\omega z\phi_4\phi_6\alpha}{r^2} \\ \frac{i\omega z\phi_3\phi_5\alpha}{r^2} & \frac{-i\omega z\phi_3\phi_4\alpha}{r^2} & 0 & \frac{i\omega z\phi_5\phi_6\alpha}{r^2} & \frac{-i\omega z\phi_4\phi_6\alpha}{r^2} & 0 \end{pmatrix}, \tag{61c}$$

with

$$f^{\pm}_{ij}(Y)=\left(\frac{Y}{r}\phi_i\pm\omega\phi_j\right)\,,\qquad \beta=\left(1-\frac{q^2z^2}{r^4}\right)\,,$$
$$g_\phi=\left(\phi_3\phi_6+\phi_1\phi_5\right)=\left(\phi_3\phi_6+\phi_2\phi_4\right)\,, \tag{61d}$$

and the abbreviations defined in Eqn. (54). Although some of these commutators are exact to all orders, i.e.

$$\left[z\overset{\star}{,}\phi_3\right]=\left[z\overset{\star}{,}\phi_6\right]=\left[\phi_3\overset{\star}{,}\phi_6\right]=0\,,\qquad \left[t\overset{\star}{,}z\right]=-i\epsilon e^{\sigma}\beta\,,$$
$$\left[t\overset{\star}{,}\phi_3\right]=i\epsilon e^{\bar{\sigma}}\frac{z\phi_3\alpha}{2r^2}\,,\qquad \left[t\overset{\star}{,}\phi_6\right]=i\epsilon e^{\bar{\sigma}}\frac{z\phi_6\alpha}{r^2}\,, \tag{62}$$

higher order corrections in other commutators and relations appear as in the Schwarzschild case above. For example

$$\begin{aligned} \phi_1 \star \phi_1 + \phi_2 \star \phi_2 &\neq \phi_3 \star \phi_3 \,, \\ \phi_4 \star \phi_4 + \phi_5 \star \phi_5 &\neq \phi_6 \star \phi_6 \,, \end{aligned} \tag{63}$$

which again could be interpreted as non-commutative correction to the embedding geometry.

4 Discussion and conclusion

In this paper, we have provided explicit realizations of the Schwarzschild and the Reissner-Nordstöm geometry as non-commutative spaces in the framework of matrix models. Our construction is based on suitable embeddings of these classical geometries $\mathcal{M}^4 \subset \mathbb{R}^D$ ("branes") in higher-dimensional flat spaces. These 4-dimensional branes are equipped with certain self-dual symplectic structures, which define the non-commutative form of these spaces via a star product. These embeddings and the corresponding symplectic structure are chosen such that they are asymptotically constant. To be more precise, for $r \to \infty$ they reduce to the usual Groenewold-Moyal quantum plane which is trivially embedded in $\mathbb{R}^D$. At the semi-classical level, the central singularity is reflected by the fact that the embedding escapes to infinity. Non-commutative effects are expected to modify this behavior, which is however not addressed in the present paper.

The requirement of asymptotic triviality is not satisfied by the standard embeddings e.g. of the Schwarzschild geometry in the literature [14–16]. This requirement is strongly suggested by the matrix model framework, because the effective action may contain terms which depend on the embedding of $\mathcal{M} \subset \mathbb{R}^D$ and not only on its intrinsic geometry. In fact, flat Groenewold-Moyal quantum planes are always solutions of this class of matrix models, independent of e.g. vacuum energy contributions. Asymptotic triviality is also natural since we want to consider our solution as a perturbation of some larger cosmological context through a localized mass. In other words, the embedding presented here should naturally generalize to many-particle configurations.

Another important aspect is that $e^{-\sigma}$, which essentially sets the scale of non-commutativity, is also asymptotically constant and non-vanishing. This must be so because $e^{-\sigma}$ determines the strength of the non-Abelian gauge coupling in the matrix model [25]. We found that $g_{\mu\nu} = G_{\mu\nu}$ (i.e. the embedding metric coincides with the effective metric, which is certainly very natural) is indeed compatible with asymptotically constant $\theta^{\mu\nu}$ and $e^{-\sigma}$. However, $e^{-\sigma}$ becomes non-trivial as one approaches the horizon. In fact, it turns out to vanish on a circle on the horizon, where the "would-be $U(1)$ gauge fields" corresponding to $\theta^{-1}_{\mu\nu}$ vanish. This result, if taken literally, is somewhat problematic from the physics point of view: if $\theta^{\mu\nu}$ is really a rigid condensate determined by its asymptotics at infinity, then the rotation on the earth with respect to such a background would lead to small variations of the gauge coupling constant during a revolution (however other quantities may also depend on $e^{-\sigma}$ and lead to cancellations of such an effect). There are stringent bounds on the variations of the fine-structure constants [26], which might exclude such an effect. If so, this would not rule out the framework, but it would strongly support the idea that the Poisson structure $\theta^{\mu\nu}$ should be integrated out resp. averaged over, rather than being a large-scale physical condensate. This is indeed very natural, since some of the degrees of freedom in $\theta^{\mu\nu}$ essentially decouple from the other fields [9]. The effective action

would then only depend on a single effective metric $G_{\mu\nu}$. This is a very attractive possibility which will be pursued elsewhere.

We also remind the reader that the appropriate equations governing $\theta^{\mu\nu}$ and therefore $e^{-\sigma}$ depend on the precise form of the action. $G_{\mu\nu} = g_{\mu\nu}$ is certainly natural and appropriate for Yang-Mills models, but was simply assumed here. Relaxing this condition might also simplify the somewhat unusual reality properties of $\theta^{\mu\nu}$. There are a lot of other obvious issues arising from our construction which deserve further studies, and the present paper should be seen as first step of a more general line of investigation. In any case, we have shown how realistic gravity can arise within this class of matrix models, in a very explicit and accessible manner. This should be enough motivation for further work.

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Appendix A: Derivation of the symplectic form Eqn. (33)

Considering Eqn. (30) we can make the additional ansatz that Θ is invariant under the Killing vector fields, i.e. $\mathcal{L}_{V_{ts}}\Theta = 0$, and moreover $\mathcal{L}_{V_{ts}}\star = \star\mathcal{L}_{V_{ts}}$. This implies $dE = 0 = \mathcal{L}_{V_{ts}}E$, and together with $\mathcal{L}_{V_\varphi}E = 0$ we obtain

$$E = E_r(r,\vartheta)dr + E_\vartheta(r,\vartheta)d\vartheta = d\chi_E(r,\vartheta)\,. \tag{64}$$

Similarly, $\mathcal{L}_{V_\varphi}\Theta = 0$ implies $\Theta_B = B \wedge d\varphi$ with

$$B = B_r d\vartheta + B_\vartheta dr = d\chi_B(r,\vartheta)\,. \tag{65}$$

Now we need to work out

$$\star\Theta = \frac{1}{2\sqrt{|g|}} g_{\alpha\alpha'} g_{\beta\beta'} \varepsilon^{\alpha'\beta'\mu\nu} \theta^{-1}_{\mu\nu} dx^\alpha \wedge dx^\beta\,, \tag{66}$$

where in Schwarzschild coordinates

$$\begin{aligned} g_{tt} &= (1 - \frac{r_c}{r}), & g_{rr} &= (1 - \frac{r_c}{r})^{-1}\,, \\ g_{\vartheta\vartheta} &= r^2, & g_{\varphi\varphi} &= r^2 \sin^2\vartheta, \\ \sqrt{|g|} &= r^2 \sin\vartheta\,. \end{aligned} \tag{67}$$

So if we define

$$\begin{aligned} \Theta_B &:= \star\Theta_E = \star(E_r dr + E_\vartheta d\vartheta) \wedge dt \\ &= \frac{1}{r^2 \sin\vartheta}\left(r^4 \sin^2\vartheta E_r d\vartheta d\varphi - r^2(1 - \frac{r_c}{r})^{-1} \sin^2\vartheta E_\vartheta dr d\varphi\right) \\ &= \sin\vartheta\left(r^2 E_r d\vartheta - (1 - \frac{r_c}{r})^{-1} E_\vartheta dr\right) \wedge d\varphi \\ &= (B_r d\vartheta + B_\vartheta dr) \wedge d\varphi\,, \end{aligned} \tag{68}$$

and if that is closed, then $\Theta = i\Theta_E + \star\Theta_E$ is self-dual. Explicitly, for

$$E = d(f(r)\cos\vartheta) = f'\cos\vartheta dr - f\sin\vartheta d\vartheta = E_r dr + E_\vartheta d\vartheta \tag{69}$$

we need

$$\begin{aligned} 0 &= d\left(r^2 E_r \sin\vartheta\, d\vartheta - (1-\frac{r_c}{r})^{-1} E_\vartheta \sin\vartheta\, dr\right) \\ &= d\left(r^2 f' \sin\vartheta\, \cos\vartheta d\vartheta + f(1-\frac{r_c}{r})^{-1}\sin^2\vartheta dr\right) \\ &= \partial_r(r^2 f')\sin\vartheta\, \cos\vartheta dr\wedge d\vartheta + 2f(1-\frac{r_c}{r})^{-1}\sin\vartheta\cos\vartheta d\vartheta\wedge dr\,, \end{aligned} \tag{70}$$

so

$$\begin{aligned} \partial_r(r^2 f') &= 2f(1-\frac{r_c}{r})^{-1}\,, \\ r^2 f'' + 2rf' - 2f(1-\frac{r_c}{r})^{-1} &= 0\,, \end{aligned} \tag{71}$$

which has the solution

$$f(r) = c_1 r(1-\frac{r_c}{r}) + c_2\frac{1}{r_c^2}\left(1-\frac{r_c}{2r} + (\frac{r}{r_c}-1)\ln(1-\frac{r_c}{r})\right)\,. \tag{72}$$

For $c_2 = 0$ we get Eqn. (33) which has the desired asymptotics as an asymptotically constant external field. Then

$$\begin{aligned} \sqrt{|\theta^{-1}|} &= \mathrm{Pfaff}(\theta^{-1}_{\mu\nu}) = \frac{1}{8}\varepsilon^{\mu\nu\rho\sigma}\theta^{-1}_{\mu\nu}\theta^{-1}_{\rho\sigma} \\ &= (\theta^{-1}_{rt}\theta^{-1}_{\vartheta\varphi} - \theta^{-1}_{\vartheta t}\theta^{-1}_{r\varphi}) \\ &= (E_r B_r - E_\vartheta B_\vartheta) \\ &= \left(f'^2 r^2\cos^2\vartheta\sin\vartheta + f^2\sin^3\vartheta(1-\frac{r_c}{r})^{-1}\right) \\ &= c_1^2 r^2 \sin\vartheta\left(1-\sin^2\vartheta\frac{r_c}{r}\right)\,, \end{aligned} \tag{73}$$

which yields (34).

Appendix B: Commutation relations for Schwarzschild geometry

From Eqn. (35) we can immediately read off $\theta^{-1}_{\mu\nu}$ in Darboux coordinates, and its inverse leads to the Poisson brackets

$$\{x_D^\mu, x_D^\nu\} = \epsilon\begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}\,, \tag{74}$$

where $\epsilon = 1/c_1$. Using the relations

$$
\begin{aligned}
H_{ts} &= r\gamma\cos\vartheta = z\left(1-\frac{r_c}{r}\right) ,\\
H_\varphi &= \frac{1}{2}r^2\sin^2\vartheta = \frac{1}{2}\left(x^2+y^2\right) ,\\
t_S &= t - r_c\ln\left|\frac{r}{r_c}-1\right| ,\\
r &= \sqrt{x^2+y^2+z^2},
\end{aligned} \tag{75}
$$

we transform the set of coordinates to $\{H_{ts}, t_S, H_\varphi, \varphi\} \to \{t, x, y, z\}$, and get

$$
\begin{aligned}
\partial_{H_{ts}} &= e^{\bar\sigma}\left(\frac{r_c z}{r\left(r-r_c\right)}\partial_t + \partial_z\right) ,\\
\partial_{t_S} &= \partial_t ,\\
\partial_{H_\varphi} &= e^{\bar\sigma}\left(\frac{r_c}{r^2}\partial_t - \frac{r_c z}{r^3}\partial_z\right) + \frac{1}{x^2+y^2}\left(x\partial_x + y\partial_y\right) ,\\
\partial_\varphi &= -y\partial_x + x\partial_y ,
\end{aligned} \tag{76}
$$

where $e^{-\bar\sigma} = \epsilon e^{-\sigma}$ is the Jacobian determinant of the transformation. We hence arrive at the following Poisson brackets in terms of the Cartesian coordinates $x^\mu = \{t, x, y, z\}$:

$$
\{x^\mu, x^\nu\} = \epsilon e^{\bar\sigma}\begin{pmatrix} 0 & -\frac{r_c y}{r^2} & \frac{r_c x}{r^2} & -i \\ \frac{r_c y}{r^2} & 0 & e^{-\bar\sigma} & -\frac{r_c y z}{r^3} \\ -\frac{r_c x}{r^2} & -e^{-\bar\sigma} & 0 & \frac{r_c x z}{r^3} \\ i & \frac{r_c y z}{r^3} & -\frac{r_c x z}{r^3} & 0 \end{pmatrix} . \tag{77}
$$

Using these, one easily works out the remaining Poisson brackets with the embedding functions ϕ_i of Eqn. (17), namely $\{x^\mu, \phi_i(x)\}$ and $\{\phi_i(x), \phi_j(x)\}$, leading finally to (38).

Next-to-leading order commutation relations: To third order in the expansion parameter ϵ one finds the star commutators

$$
\begin{aligned}
[t \stackrel{\star}{,} x] &= -i\epsilon\frac{r_c e^{\bar\sigma}}{r^2}y - \epsilon^3 y F_{txy} + \mathcal{O}\left(\epsilon^5\right) ,\\
[t \stackrel{\star}{,} y] &= i\epsilon\frac{r_c e^{\bar\sigma}}{r^2}x + \epsilon^3 x F_{txy} + \mathcal{O}\left(\epsilon^5\right) ,\\
[t \stackrel{\star}{,} z] &= \epsilon e^{\bar\sigma} ,\\
[x \stackrel{\star}{,} y] &= i\epsilon ,\\
[x \stackrel{\star}{,} z] &= -i\epsilon y\frac{r_c z e^{\bar\sigma}}{r^3} - \epsilon^3 y F_{zxy} + \mathcal{O}\left(\epsilon^5\right) ,\\
[y \stackrel{\star}{,} z] &= -i\epsilon x\frac{r_c z e^{\bar\sigma}}{r^3} + \epsilon^3 x F_{zxy} + \mathcal{O}\left(\epsilon^5\right) ,
\end{aligned} \tag{78}
$$

with the abbreviations

$$\begin{aligned}
F_{txy} &= \frac{r_c e^{5\bar\sigma}}{24 r^{14}}\Big(\gamma^2 r^6(3r_c^2 - 9r_c r + 8r^2) - \gamma r_c r^4(6\gamma r_c + 17r)z^2 \\
&\qquad + r_c^2(3r_c^2 + 3r_c r + 2r^2)z^4\Big)\,, \\
F_{zxy} &= \frac{r_c z e^{5\bar\sigma}}{8 r^{15}}\Big(\gamma^2 r^6(r_c^2 - 4r_c r + 5r^2) + 2\gamma(\gamma^2-3)r_c r^5 z^2 + r_c^4 z^4\Big)\,. \qquad (79)
\end{aligned}$$

Notice, that some expressions (i.e. the ones where $\mathcal{O}(\epsilon^n)$ is omitted) are exact to all orders[4]. Furthermore, we find for the embedding functions ϕ_i to third order in ϵ:

$$\begin{aligned}
[t \stackrel{\star}{,} \phi_1] &= -\epsilon e^{\bar\sigma}\frac{z}{r}f_{12}^{+}(0) + \mathcal{O}(\epsilon^5) = -\epsilon e^{\bar\sigma}\frac{z}{r}\left(\frac{\phi_1}{2r} + \omega\phi_2\right) + \epsilon^3\phi_2 F_{t\phi 12} + \mathcal{O}(\epsilon^5)\,, \\
[t \stackrel{\star}{,} \phi_2] &= -\epsilon e^{\bar\sigma}\frac{z}{r}f_{21}^{-}(0) + \mathcal{O}(\epsilon^5) = -\epsilon e^{\bar\sigma}\frac{z}{r}\left(\frac{\phi_2}{2r} - \omega\phi_1\right) - \epsilon^3\phi_1 F_{t\phi 12} + \mathcal{O}(\epsilon^5)\,, \\
[t \stackrel{\star}{,} \phi_3] &= -\epsilon e^{\bar\sigma}\frac{z\phi_3}{2r^2}\,, \\
[x \stackrel{\star}{,} \phi_1] &= -i\epsilon e^{\bar\sigma}\frac{y}{r}f_{12}^{+}(r_c) + \epsilon^3 y F_{xy}(\phi_1,\phi_2) + \mathcal{O}(\epsilon^5)\,, \\
[x \stackrel{\star}{,} \phi_2] &= -i\epsilon e^{\bar\sigma}\frac{y}{r}f_{21}^{-}(r_c) + \epsilon^3 y F_{xy}(-\phi_2,\phi_1) + \mathcal{O}(\epsilon^5)\,, \\
[x \stackrel{\star}{,} \phi_3] &= -i\epsilon e^{\bar\sigma}\frac{y\gamma\phi_3}{2r^2} + \epsilon^3 y F_{\phi 3xy} + \mathcal{O}(\epsilon^5)\,, \\
[y \stackrel{\star}{,} \phi_1] &= i\epsilon e^{\bar\sigma}\frac{x}{r}f_{12}^{+}(r_c) - \epsilon^3 x F_{xy}(\phi_1,\phi_2) + \mathcal{O}(\epsilon^5)\,, \\
[y \stackrel{\star}{,} \phi_2] &= i\epsilon e^{\bar\sigma}\frac{x}{r}f_{21}^{-}(r_c) - \epsilon^3 x F_{xy}(-\phi_2,\phi_1) + \mathcal{O}(\epsilon^5)\,, \\
[y \stackrel{\star}{,} \phi_3] &= i\epsilon e^{\bar\sigma}\frac{x\gamma\phi_3}{2r^2} - \epsilon^3 x F_{\phi 3xy} + \mathcal{O}(\epsilon^5)\,, \\
[z \stackrel{\star}{,} \phi_1] &= \epsilon e^{\bar\sigma}\omega\phi_2 + \epsilon^3\phi_2 F_{z\phi 12} + \mathcal{O}(\epsilon^5)\,, \\
[z \stackrel{\star}{,} \phi_2] &= -\epsilon e^{\bar\sigma}\omega\phi_1 - \epsilon^3\phi_1 F_{z\phi 12} + \mathcal{O}(\epsilon^5)\,, \\
[z \stackrel{\star}{,} \phi_3] &= 0\,, \qquad (80)
\end{aligned}$$

with

$$\begin{aligned}
F_{t\phi 12} &= \frac{r_c z\omega^3 e^{2\bar\sigma}}{24\gamma^3 r^6}\Big(27e^{3\bar\sigma}(z^2 + r^2\gamma(1+\gamma)) - 6\gamma r^2 + e^{\bar\sigma}(3(7+\gamma(9+5\gamma))r^2 + 2z^2) \\
&\qquad - 3e^{2\bar\sigma}\left((9+2\gamma(8+7\gamma))r^2 + 7z^2\right)\Big)\,, \\
F_{\phi 3xy} &= ie^{5\bar\sigma}\frac{\gamma\phi_3}{64 r^{10}}\Big(15\gamma^4 r^4 - 2\gamma\left(8+\gamma\left(9+\gamma(15\gamma-32)\right)\right)r^2 z^2 \\
&\qquad + (1-\gamma)^2\left(20+\gamma(15\gamma-34)\right)z^4\Big)\,,
\end{aligned}$$

[4]However, while this is the case for the commutator $[x \stackrel{\star}{,} y] = i\epsilon$, the according anticommutator does in fact have higher order contributions, i.e.

$$[x \stackrel{\star}{,} y]_+ = 2xy - xy\left(\frac{\epsilon^2}{(x^2+y^2)^2} - \frac{\epsilon^4}{4(x^2+y^2)^4}\right) + \mathcal{O}(\epsilon^6)\,.$$

$$F_{z\phi 12} = \frac{\omega^3 e^{3\bar\sigma}}{8r^4}(e^{\bar\sigma}-1)\Big((9e^{\bar\sigma}-4)z^2 - r^2\Big)\,, \tag{81}$$

and

$$\begin{aligned} F_{xy}(\phi_i,\phi_j) = &-\frac{ie^{5\bar\sigma}}{192r^{14}}\Big(8e^{-2\bar\sigma}\omega^3\phi_j r^{11} - 12\omega^2\phi_i r^3\left[(r_c-3r)\gamma^2 r^6 + 2r_c^2 z^2\gamma r^3 + r_c^2(r_c+3r)z^4\right] \\ &- 2\omega\phi_j r^4\left[\left(15r_c^2 - 40rr_c + 33r^2\right)\gamma^2 r^3 + 2r_c\left(15r_c^2 - 16rr_c - 33r^2\right)z^2\gamma + r_c^2\left(15r_c^2 + 8rr_c + 9r^2\right)\tfrac{z^4}{r^3}\right] \\ &- 3\gamma\phi_i\left[15\gamma^4 r^8 + 2r_c\left(15r_c^2 - 13rr_c - 10r^2\right)z^2\gamma r^3 + r_c^2\left(15r_c^2 + 4rr_c + r^2\right)z^4\right]\Big)\,. \end{aligned} \tag{82}$$

Finally we also have

$$\begin{aligned} [\phi_1 \stackrel{\star}{,} \phi_2] &= \epsilon e^{\bar\sigma}\frac{\omega z\phi_3^2}{2r^2} + \epsilon^3 e^{3\bar\sigma}\frac{\omega^3 z\phi_3^2}{16r^6}\Big(4e^{\bar\sigma}(r^2+z^2) - r^2 - 9z^2e^{2\bar\sigma}\Big) + \mathcal{O}\left(\epsilon^5\right), \\ [\phi_1 \stackrel{\star}{,} \phi_3] &= \epsilon e^{\bar\sigma}\frac{\omega z\phi_3\phi_2}{2r^2} + \epsilon^3\phi_2 F_{\phi 312} + \mathcal{O}\left(\epsilon^5\right)\;, \\ [\phi_2 \stackrel{\star}{,} \phi_3] &= -\epsilon e^{\bar\sigma}\frac{\omega z\phi_3\phi_1}{2r^2} - \epsilon^3\phi_1 F_{\phi 312} + \mathcal{O}\left(\epsilon^5\right)\;, \end{aligned} \tag{83}$$

with

$$F_{\phi 312} = e^{3\bar\sigma}\frac{\omega^3 z\phi_3}{64r^6}\Big((1 - 22e^{\bar\sigma} + 36e^{2\bar\sigma})(x^2+y^2) - (7 - 38e^{\bar\sigma} + 36e^{2\bar\sigma})r^2\Big)\,. \tag{84}$$

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